Max Flows 3: Preflow-Push

Wednesday, 6 October 2021 2:00 PM

MAX FLOWS : THE PREFLOW-PUSH / PUSH-RELABEL ALGORITHM

E-K & F-F	P-P
- maintains a flow in execution	- maintains a preflow f at every iteration
- Stops when no s-t path in residual graph	- maintains no s-t path, stopping criterion: f is a flow.

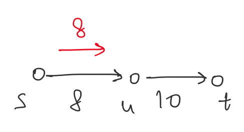
Given : $G=(V,E)$, c_e, $s,t \in V$

Preflow : $f: E \to \mathbb{R}_+$ (actually s-t)

(i) $\forall e \quad f(e) \le c_e$

(ii) $excess_f(v) = \sum_{e \text{ into } v} f(e) - \sum_{e \text{ out of } v} f(e)$

$\forall v \ne s, \quad excess_f(v) \ge 0$



This is a preflow.

Thus for any preflow f, $excess_f(s) \le 0$

Algorithm:

Initially : $\forall e=(s,v)$ $f(e)=c_e$, $f(e)=0$ for other edges

$\ell(s)=n$, $\ell(v)=0$ for all $v \ne s$ (n: # vertices)

while $\exists v \ne t$: $excess_f(v) > 0$

  if $\exists u : (v,u) \in E_f$ and $\ell(u) = \ell(v)-1$

    $\delta = \min\{excess_f(v),\ c^f(v,u)\}$

    if (v,u) is a fwd edge, $f(v,u) = f(v,u)+\delta$

    else $f(u,v) = f(u,v) - \delta$

  else $\ell(v) \leftarrow \ell(v)+1$

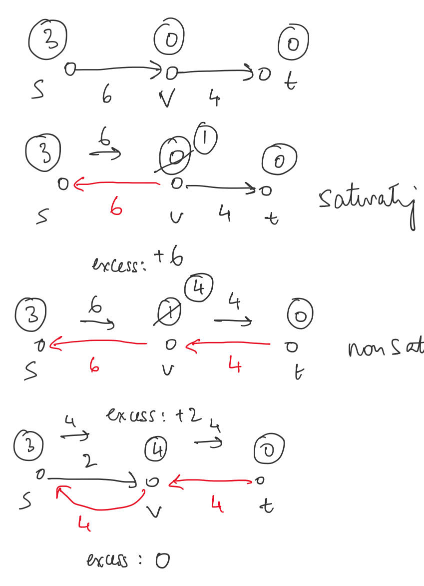
Observations : (i) labels are monotone non-decreasing

(ii) f is always a preflow

(iii) $\sum_v excess_f(v) = 0$, & $\forall v \ne s$, $excess_f(v) \ge 0$

$\Rightarrow excess_f(s) \le 0 \Rightarrow \ell(s)$ is unchanged.

Example:

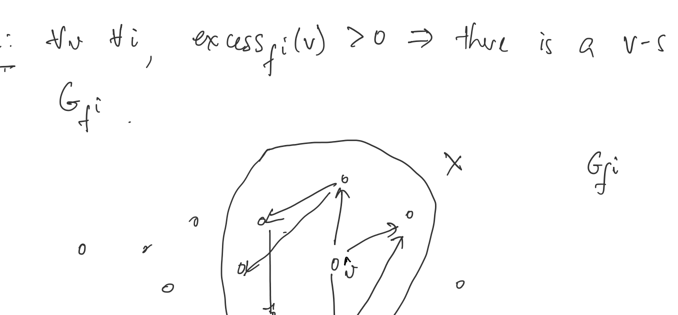


Say that f^i is the preflow after i^{th} iteration of the while loop.

$$f^0_e = \begin{cases} c_e & \text{is } e=(s,v) \\ 0 & \text{o.w.} \end{cases}$$

Claim 1: $\forall v\ \forall i$, $excess_{f^i}(v) > 0 \Rightarrow$ there is a v-s path in G_{f^i}.

Proof :



Fix some $\hat{v}, i$ s.t. $excess_{f^i}(\hat{v}) > 0$.

Say X is set of vertices reachable from $\hat{v}$ in G_{f^i}.

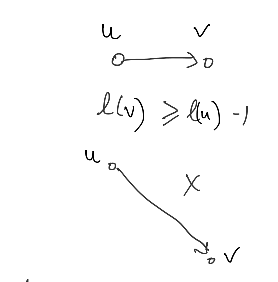
Thus $\sum_{u \in X} excess_{f^i}(u) \le 0$ —— (1) (verify yourself) (expand sum, reverse order of summation)

However $\hat{v} \in X$, $excess_{f^i}(v) > 0$. Further $\forall u \ne s$, $excess_{f^i}(u) \ge 0$

Hence $s \in X$ (since only s can have -ve excess) ∎

Claim 2 : $\forall i$, if $(u,v) \in G_{f^i}$, then

$\ell(v) \ge \ell(u) - 1$



Proof : By induction on i.

Consider f^0.

(i) $\forall v \ne s$, $\ell(v) = 0$. Thus $\forall e=(u,v)$ where $u,v \ne s$ this is satisfied.

(ii) all edges incident on s are coming into s

$s \leftarrow u$

$\ell(s) = n \ge \ell(u)$ since $\ell(u)=0$

Hence initially this condition is satisfied.

Suppose true for f^{i-1}

Case I: i^{th} operation is relabel, say $\ell(u) \leftarrow \ell(u)+1$

$u \leftarrow v$ these edges are fine

$u \rightarrow v$ Consider some (u,v) edge in G_{f^i} ($= G_{f^{i-1}}$). Hence $(u,v) \in E_{f^{i-1}}$.

By induction $\ell(v) \ge \ell(u) - 1$ earlier.

However, since we are doing relabel opn.,

$excess_{f^{i-1}}(u) > 0$, & $\ell(v) \ne \ell(u)-1 \Rightarrow \ell(v) \ge \ell(u)$ earlier (before relabel opn.)

Hence after relabel opn., $\ell(v) > \ell(u)$.

Case II: i^{th} operation is push opn.

Say we push along (v,u) edge. Then $\ell(u) = \ell(v) - 1$

Diff between $E_{f^{i-1}}$ & E_{f^i} is that there could be an addl. (u,v) edge in E_{f^i}. But $\ell(v) = \ell(u)+1 \ge \ell(u) - 1$. Hence claim holds. ∎

Corollary 1 : For any i, G_{f^i} does not have an s-t path. Thus when algorithm terminates, f is a max flow.

Corollary 2 : $\forall v, \forall i$, $\ell(v) \le 2n$

Proof : Suppose $\exists \hat{v}, i$ s.t. $\ell(\hat{v}) = 2n$ & $excess_{f^{i-1}}(\hat{v}) > 0$.

Then by Claim 1, $G_{f^{i-1}}$ has a $\hat{v}$-s path of length $\le n-1$

But by Claim 2, on every edge of this path, label decreases by at most 1. Hence $\ell(s) \ge n+1$ which is a contradiction ∎

Corollary 3: # of relabel opns. is at most $2n^2$

Now we need to bound # push opns.

2 diff kinds of push opns.: along (v,u) edge

- $\delta = excess_f(v)$ non-saturating push
- $\delta = c^f(v,u)$ saturating push

Consider saturating push along (v,u) edge.

This is similar to E-K, where whenever an edge (v,u) reappears, dist(v) increases by 2

Now for P-P, similarly, (i) edge (v,u) disappears, & (ii) whenever edge (v,u) reappears, $\ell(v)$ increases by 2.

(prove yourself)

For any edge (v,u), # saturating pushes $\le n$ (since $\ell(v) \le 2n$).

Hence # saturating pushes $\le 2mn$

Lastly, to obtain bound on # non-saturating pushes.

Consider a N-S push along (v,u). Then $excess_f(v)$ goes to 0.

Intuition : labels of vertices that have excess flow gets lower & lower.

Consider :

$$\phi(i) = \sum_{v:\, excess_{f^i}(v) > 0} \ell(v) \quad . \text{ Initially, } \phi(0) = 0$$

If opn. is a non-saturating push.

then ϕ decreases by at least 1. (could be more than 1 also)

If opn. is a saturating push., then $excess_f(v)$ might not go to zero. In this case, ϕ might increase.

However, increase $\le 2n$

If opn. is a relabel, then ϕ increases exactly by 1.

Therefore # non-saturating pushes $\le$ total increase in ϕ

$\le 1 . 2n^2 + 2n . 2mn$

$= 2n^2 + 4mn^2$

Thus algorithm terminates in a max-flow in $O(mn^2)$ iteration

With appropriate data structures, can run in $O(mn^2)$ time.

With careful choice of excess vtx. in while loop, can run in $O(n^2\sqrt{m})$ time.